Free Vibration $\rightarrow$ $F(t) = 0$ $\rightarrow$ $m\ddot{x} + c\dot{x} + kx = 0$

1 DOF system $\rightarrow$ 2nd order ODE $\rightarrow$ Need 2 initial conditions to solve

$x(0) = x_0$ initial displacement

$\dot{x}(0) = \dot{x}_0 = v_0$ initial velocity

Undamped System : (no $c\dot{x}$ term)

$$m\ddot{x} + kx = 0$$

Assuming system vibrates harmonically, $x = A(\sin\omega t + \varphi)$

$\rightarrow$ $\ddot{x} = -\omega^2 A\sin(\omega t + \varphi)$

substituting into ODE :

$(-\omega^2 m + k)A\sin(\omega t + \varphi) = 0$ 'lhs of equation = 0 for all time'

sin $\neq 0$ for all time

$-\omega^2 m + k = 0$ $\rightarrow$ $\boxed{\omega_0^2 = \frac{k}{m}}$ Natural frequency, ω_0

$\rightarrow$ $\boxed{f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}}$

Considering general initial conditions :

$x(0) = x_0 = A\sin(\omega_0(0) + \varphi) = A\sin(\varphi)$

$\dot{x}(0) = v_0 = \omega_0 A\cos(\omega(0) + \varphi) = \omega_0 A\cos(\varphi)$

$\rightarrow$ $A = \sqrt{x_0^2 + v_0^2/\omega_0^2}$ & $\varphi = \tan^{-1}(x_0\omega_0 / v_0)$

$$\boxed{x = A\sin(\omega_0 t + \varphi)}$$

after converting equation of motion to standard form, we can divide by the 2nd order coefficient :

$$m\ddot{x} + kx = 0$$

÷ by 2nd order coefficient : $\ddot{x} + \frac{k}{m}x = 0$

substitute ω_0 : $\boxed{\ddot{x} + \omega_0^2 x = 0}$

whatever coefficient is here with eqn. in this form $= \omega_0^2$

Damped Vibration :

EOF for damped system includes damping term :

$$m\ddot{x} + c\dot{x} + kx = 0$$

We will use trial solution of $\boxed{x(t) = Ae^{st} = Ae^{s_R t}e^{is_i t}}$ s is a complex number

$$\dot{x} = sAe^{st}$$
$$\ddot{x} = s^2Ae^{st}$$

Substituting solution into ODE : $(ms^2 + cs + k)Ae^{st} = 0$

$\neq 0$

$$(ms^2 + cs + k) = 0$$

$$\therefore \quad s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

two real solutions
two complex solution
one (double) real solution

if we have 2 distinct roots $(s_1 \neq s_2)$:

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Rearranging $s_{1,2}$ into more convinient form :

$$s_{1,2} = -\frac{c}{2m} \pm \boxed{\sqrt{\frac{k}{m}}}\left(\boxed{\frac{c^2}{4mk} - 1}\right)^{1/2}$$

sign of this expression determines type of solution

$c^2/4mk > 1 \longrightarrow$ two distinct real roots $\longrightarrow$ over-damped system

$c^2/4mk < 1 \longrightarrow$ two complex conjugate roots $\longrightarrow$ under-damped system

$c^2/4mk = 0 \longrightarrow$ one double real root ($s_1 = s_2$) $\rightarrow$ critically damped

$C_{cr} = 2\sqrt{mk} \rightarrow$ defines boundary between oscillatory and non-oscillatory behaviour.

critical damping

$$\boxed{\frac{c}{c_{cr}} = \frac{c}{2\sqrt{mk}} = \zeta}$$, damping ratio

Can rewrite solution to $\boxed{s_{1,2} = -\zeta\omega_0 \pm \omega_0(\zeta^2-1)^{1/2}}$

For underdamped motion, $0 < \zeta < 1 \rightarrow \boxed{s_{1,2} = -\zeta\omega_0 \pm i\omega_0\sqrt{1-\zeta^2}}$

Damped natural freq, $\boxed{\omega_D = \omega_0\sqrt{1-\zeta^2}}$

$$x(t) = A_1 e^{(-\zeta\omega_0 + i\omega_D)t} + A_2 e^{(-\zeta\omega_0 - i\omega_D)t} = \boxed{Ae^{-\zeta\omega_0 t}\cos(\omega_D t - \phi)}$$

exponential decay due to damping

harmonic motion with freq. ω_D

phase lag due to damping

For overdamped motion, $\zeta > 1 \rightarrow s_{1,2} = \zeta\omega_0 \pm \omega_0\sqrt{\zeta^2-1}$

$$\boxed{x(t) = e^{-\zeta\omega_0 t}\left(A_1 e^{\omega_0\sqrt{\zeta^2-1}\,t} + A_2 e^{-\omega_0\sqrt{\zeta^2-1}\,t}\right)} \rightarrow \text{APERIODIC}$$

exponential decay due to damping

real non-oscillatory motion

For critically damped motion, $\zeta = 1 \longrightarrow s_{1,2} = -\omega_0$

$$\boxed{x(t) = (A_1 + tA_2)e^{-\omega_0 t}}$$

exponentially decaying response, no oscillation

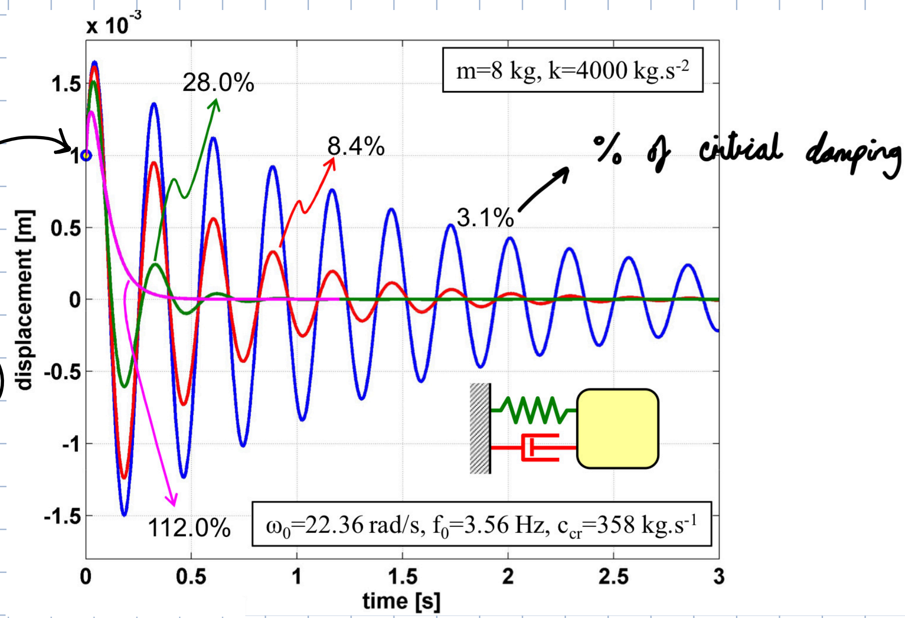
Substituting new relationships for ω_0 & ζ into standard EoM :

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \longrightarrow \ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0$$

Same initial conditions

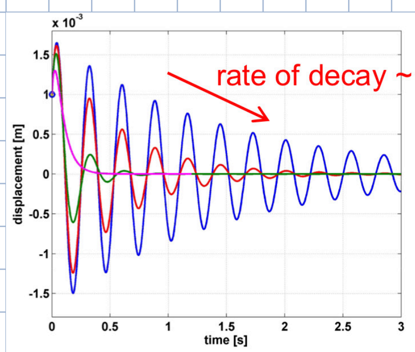
→ same x_0 (displacement = 1)

→ same V_0 (same initial slope)



% of critical damping

Experimental Identification of Damping :



t_1 usually peak response

$$\frac{x(t_1)}{x(t_1 + T_D)} = damping\,?$$

damped natural period

↳ peak to peak

$$\frac{x(t)}{x(t_1+T_D)} = \frac{Xe^{-\zeta\omega_0 t_1}\sin(\omega_0 t_1 + \varphi)}{Xe^{-\zeta\omega_0(t_1+T_D)}\sin(\omega_0(t_1+T_D)+\varphi)} = e^{\zeta\omega_0 T_D}$$

$$\Lambda = \ln\left(\frac{x(t)}{x(t_1+T_D)}\right) = \zeta\omega_0 T_D = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \approx 2\pi\zeta$$

logarithmic decrement

assuming $\zeta << 1$

$$\boxed{\zeta_{exp} \approx \frac{\Lambda}{2\pi}}$$

experimentally found ζ

We can also consider two displacements seperated by N periods T_0

→ could be very small difference between adjacent peaks, & noise may impact accuracy. Using more periods gives greater difference.

$$\boxed{\zeta_{exp} \approx \frac{1}{2\pi N} \ln \frac{x(t)}{x(t_1 + NT_0)}}$$